How Stars Work

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Preface

We live around a star, the Sun. It is overwhelmingly the most important feature of our environment – too bright to look at! – and drives the dynamics of our climate, our economy, and our daily routine. We understand it in remarkable detail: we can measure its internal sounds and buoyancy waves as it rings like a bell, and using the science of helioseismology we can accurately reconstruct its physical properties almost all the way to the core. We can see its magnetised wind as the Southern Lights, and measure it quantitatively on satellites. We we can even measure neutrinos directly reaching us from the nuclear furnace at its core.

More broadly, stars are to astronomy what atoms are to chemistry: innumerable building blocks of a much larger universe of galaxies and cosmology. Each has remarkably simple physics, really parametrized to a very good approximation only by their mass and composition, and at higher order their rotation period; these physics are so simple and reliable that we can accurately model the spectra of whole galaxies in the distant universe just using laboratory physics and our understanding of stars nearby.

Analytic reasoning is usually good enough to get rough understanding of the important phenomena in stellar physics, usually as scaling relations - but these are usually only accurate to an order of magnitude or so, and *real* models require computer calculations. In this course we will attempt to do a bit of both.

This is a series of notes, aimed at a second-year undergraduate level, about how stars work: exploring their simplicity and complexity, assuming only a first year undergrad level of classical mechanics, quantum mechanics, and thermal physics, and other physics that we will introduce *ad hoc* as we go along.

These notes are based in part on the excellent lectures from which I originally learned as an undergraduate at Berkeley, by Eliot Quataert; on Peter Tuthill and Mike Ireland's courses at Sydney; and on the PHYS2082 course developed at UQ by Holger Baumgardt, and by myself.

This book is compiled using Jupyter-Books, which allows us to include Python calculations in the text; I aim to make this available in the Open Astrophysics Library when it is in a mature state. We will follow the style guide of Edward Tufte, using margins for asides and illustrations to avoid interrupting the flow of the text.

We will use *cgs* units for most calculations in this text, except where otherwise noted.

Notes

We will be using Python for computation in this book: in every case, we will be using the same fundamental constants in SI units, and the same imports of basic Python libraries:

```
import numpy as np
import matplotlib.pyplot as plt

G = 6.6743e-11 # N m2 kg-2; Newton's Constant
k_B = 1.380649e-23 # J/K; Boltzmann's Constant
m_p = 1.67262192e-27 # kg; proton mass

M_sun = 1.9884e30 # kg
R_sun = 6.957e8 # m
Teff_sun = 5780 # effective temperature, K

M_earth = 5.972e24 # kg
R_earth = 6371e3 # m
```

1 Hydrostatic Equilibrium

The fundamental physics of stars is determined by a handful of principles:

- the star is everywhere in pressure balance under its own gravity, or *hydrostatic equilibrium*;
- its cooling by radiation must be met by
  - *energy production* in its interior by nuclear fusion or gravitational contraction,
  - and *energy transport* to its surface by radiation, convection, and conduction;
- how its material responds to pressure (parametrized the *equation of state*) and to light (parametrized by *opacity*); and
- in its interior and a star is typically rotating and magnetic, which we will neglect in most of these notes.

hydrostatic from Greek ὕδωϱ, 'water', and στάσις, 'standing'; just like water in a tank.

Let's talk about hydrostatic equilibrium first.

An ordinary star like the Sun, throughout its whole body, is to a very good approximation an *ideal gas*, and is fully ionized except in its outermost layer. This means that the gas pressure satisfies the equation of state

$$p = nk_B T$$

where p is the pressure, n is the number density (particles per volume) of the gas, k_B is Boltzmann's constant (1.38×10^{-16} erg/K), and T is the temperature in kelvin.

In a star like the Sun, the pressure is mostly provided by gas pressure. In hotter stars, *photon* or *radiation pressure* is dominant, but in the Sun this is $\sim 10^{-3} p_{\text{gas}}$.

Even in the Sun, though, the gas is not *quite* a classical ideal gas: quantum mechanics is already relevant. There is an equation we will derive later in these notes for the pressure due to the *degeneracy* of a gas where the quantum wavefunctions of its constituent particles are nearly overlapping:

$$p_{\text{degeneracy}} = \frac{\hbar}{5m_e}(\frac{3}{8\pi})^{2/3}n^{5/3}$$

It turns out this is about a quarter of the gas pressure at the core of the Sun!

where $\hbar$ is the quantum of action $h/2\pi$ $(1.0546 \times 10^{-27}\text{erg} \cdot \text{s})$, and m_e the mass of the electron.

1.1 The Equation of Hydrostatic Equilibrium

Consider a thin shell of radius r (and surface area $A = 4\pi r^2$), thickness dr, and mass density ρ, enclosing a mass M_r.

i.e. M_r is the total mass integrated out up to a radius r.

The mass of this shell is $M_{\text{shell}} = \rho A dr$, and from Newton's law of gravitation the magnitude of the gravitational force of the whole shell inwards is

Newton's Shell Theorem states that the gravitational attraction of a symmetric shell of matter, and therefore by linearity of a ball of matter, can be treated as if the mass were concentrated at a point at the centre.

$$\frac{-GM_r M_{\text{shell}}}{r^2}$$

So now we can calculate the net force on this shell (and therefore acceleration a), and require the forces to be in balance:

$$F_{\text{net}} = M_{\text{shell}}a = P_{\text{below}} \cdot A - P_{\text{above}} \cdot A - \frac{GM_r M_{\text{shell}}}{r^2}$$

Letting $P_{\text{above}} = P_{\text{below}} + dP$,

$$M_{\text{shell}}a = a\rho A dr = -AdP - \frac{GM_r M_{\text{shell}}}{r^2}$$

and therefore rearranging, in equilibrium ($a = 0$) we have the **equation of hydrostatic equilibrium**:

$$\frac{dP}{dr} = -\rho\frac{GM_r}{r^2} \tag{1.1}$$

Tattoo this equation on the back of your eyelids.

1.2 Plane-Parallel Approximation

This is actually a very familiar equation that we encounter not just in first-year physics, but in everyday life. Nearly all the mass of the Earth is in its body, and only a bit less than a millionth in its atmosphere. So we can consider a situation where

$$r = R_{\oplus} + z$$

where z is the height in the atmosphere $\lll R_{\oplus}$. We often denote astronomical bodies by traditional symbols: $\oplus$ is the astronomical symbol for the Earth, and $\odot$ the Sun. There are many other traditional symbols that are now rarely used.

We can now write this in familiar terms with the acceleration due to gravity as

$$\frac{dP}{dh} = -\rho \cdot \underbrace{\frac{GM_r}{r^2}}_{\equiv g, \text{ constant}}$$

If we are dealing with an incompressible liquid like water, then ρ is a constant and we simply have

$$P(z) = P_0 - \rho g z$$

We can rearrange this to solve for the depth of water required to reach a gauge pressure $\Delta P \equiv P - P_0$:

$$z = -\frac{\Delta P}{\rho g}$$

For $\rho_{\text{water}} = 1000 \text{kg m}^{-3}$, to get a 1 atmosphere ($\approx 10^6$ Pa) gauge pressure requires 10.34 m of water.

Things are different when your density depends on pressure. For an ideal gas, $P = nk_BT$ and $\rho = n\langle m \rangle$ - so we have an *equation of state* that we can use to solve for the vertical structure of an isothermal atmosphere:

where $\langle m \rangle$ is the mean molecular mass, and

isothermal means having the same temperature everywhere, from ἴσος, “same”, and θέρμη, “heat”.

$$\frac{d}{dz}(nk_BT) = -\langle m \rangle g n$$

$$\frac{dn}{dz} = -\frac{\langle m \rangle g}{k_BT} \cdot N$$

$$n = n_0 \exp{(-\frac{\langle m \rangle g}{k_BT} z)}$$

$$n = n_0 \exp{-z/h}$$

and we call h the scale height in the atmosphere. Let’s calculate this for some interesting situations!

First let's try an isothermal Earth atmosphere:

```
mol_earth = 28.964 * m_p # average molecular weight for earth's atmosphere
T_earth = 300 # K; room temperature
g_earth = 9.8 # m/s

h = k_B * T_earth / mol_earth / g_earth
```

This scale height of 8.7 km is $1.4 \times 10^{-3} R_{\oplus}$; the atmosphere really is very thin and can be treated as plane-parallel. This scale height would mean that the air is 96% as dense at the top of the Q1 Tower, and 36% as dense at the summit of Mt Everest.

Q1 Tower on the Gold Coast is the tallest tower in Australia, at 322.5 m; Mt Everest, Earth's tallest mountain, stands 8849 m tall.

Now let's plug in some numbers for the Sun, using Python:

```
m = 0.5 * m_p # kg
# mean molecular weight for ionized hydrogen =
# mean of electron & proton = m_p/2

g = G * M_sun / R_sun**2
h = k_B * Teff_sun / m / g
```

giving a gravity $g = 28.0$ Earth gravities, and a scale height of $5.0 \times 10^{-4} R_{\odot}$. So we see that the solar atmosphere has a *tiny* scale height relative to the overall size of the Sun, and the plane-parallel approximation is even better than on Earth!

1.3 Scaling Relations

Most of the time, it is not possible to calculate properties of realistic stars in closed-form equations, and we will have to use computer models. But what *understanding* does this buy us? It is often just as important to have a sense of how these properties scale with one another in general terms, even if we might not be able to estimate actual numbers to better than an order of magnitude.

As a first example, let's estimate the pressure at the centre of the Sun. Ideally, we would solve for the full stellar structure using some equation of state, and then integrate $\frac{dP}{dr}$ from zero at the surface $r = R_\odot$ all the way to the core. Instead, we're going to do something very handwavy. Let's rearrange the $\frac{dP}{dr}$ in Equation 1.1 to instead be the mean gradient $P/R_\odot$:

$$\begin{aligned}
\frac{dP}{dr} &= -\rho \frac{GM_r}{r^2} \\
\frac{P}{R_\odot} &\approx \langle\rho\rangle \frac{GM_\odot}{R_\odot^2} \\
P &\approx \langle\rho\rangle \frac{GM_\odot}{R_\odot} \\
&\approx \frac{M_\odot}{4/3\pi R_\odot^3} \frac{GM_\odot}{R_\odot} \\
&\approx \frac{GM_\odot^2}{4/3\pi R_\odot^4} \qquad \propto \frac{GM_\odot^2}{R_\odot^4}
\end{aligned}$$

Remarkably, it can be shown analytically (Milne 1936) that $P > \frac{GM^2}{8\pi R^4}$ for an arbitrary star - so we are not even *too* far from the exact solution with these mathematical sins. It is often the case that very coarse approximations differ from exact solutions by a factor of order unity; we will therefore often drop these factors and just think about the scaling.

which for the Sun, gives 1.1×10^{15} Pa pressure, which is within an order of magnitude of the central pressure of 2.3×10^{16} Pa in the Standard Solar Model (Guenther et al. 1992).

We are going to do this kind of approximation a lot in these notes: to estimate how nuclear reactions scale with temperature, how stellar luminosities depend on mass, how stars form and how they die - and we will also tackle real computer models of these processes, so that we build both intuitive understanding and accurate models we can fit to data. Both are essential tools in astronomy.

"May God us keep From Single vision & Newtons sleep." —William Blake, Letter to Thomas Butts, 22 November 1802.

2 Timescales in Stars

We can be fairly sure that we are in hydrostatic equilibrium in a star, because we can calculate the timescale for variations in the pressure in a star to propagate through its body.

References

Guenther, D. B., P. Demarque, Y. -C. Kim, and M. H. Pinsonneault. 1992. "Standard Solar Model" 387 (March): 372. https://doi.org/10.1086/171090.

Milne, E. A. 1936. "The Pressure in the interior of a star." *MNRAS* 96 (January): 179. https://doi.org/10.1093/mnras/96.3.179.